Invariant measures for a stochastic Kuramoto-Sivashinky equation

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Abstract: For the 1-dimensional Kuramoto–Sivashinsky equation with random forcing term, existence and uniqueness of solutions is proved. Then, the Markovian semigroup is well defined; its properties are analyzed, in order to provide sufficient conditions for existence and uniqueness of invariant measures for this stochastic equation. Finally, regularity results are presented.

Keywords: pathwise uniqueness, invariant measures, irreducibility, strongly Feller.

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1 INTRODUCTION

We consider the 1-dimensional Kuramoto–Sivashinsky equation perturbed by an additive noise:

$$du(t,x) + [\nu u_{xxxx}(t,x) + u_{xx}(t,x) + u(t,x)u_x(t,x)]\ dt = dW(t,x) \tag{1.1}$$

where $t \in [0,T]$, $x \in \mathbb{R}$; $\nu > 0$ is a given coefficient. By u_x, u_{xx}, u_{xxxx} we denote, respectively, the first, second and fourth derivative of u with respect to the space variable x. Periodic conditions (with period L) are assumed and an initial data $u(0,x)$ is assigned. In the right hand side of equation (1.1) there is a Wiener process W with covariance $\mathbb{E}[W(t,x)W(t',x')] = (t \wedge t')q(x,x')$.

This stochastic equation is presented in the physical literature (see [1, 2, 3] and references therein) in relation to a model for erosion by ion sputtering and has been studied from the mathematical point of view in [4] and [5].
The deterministic Kuramoto–Sivashinsky equation, i.e. equation (1.1) without noise, has been introduced in [6, 7, 8, 9] in the 70's and from that time has attracted the interest of many mathematicians (see, e.g., [10] for the basic results, and the references therein). It is known that it has a finite-dimensional maximal attractor and an inertial manifold. Numerical studies show chaotic behavior of

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its solutions, see [6]. However, it may happen that the dynamics has a more regular behavior as far as statistical quantities, i.e. ensemble averages, are involved. This happens, for instance, in fluid dynamics: the individual solutions are chaotic but statistical properties of the dynamics are more regular, as investigated by turbulence theory. The results on invariant measures, presented in this paper for the stochastic Kuramoto–Sivashinsky equation (1.1), are exactly about the statistical behavior of the solution.

Comparing our results with those of [4] and [5], we notice that we construct solutions to equation (1.1) under assumptions on the noise weaker than in [4] and [5] so to analyze the case presented in [1] and [2]; moreover we deal with invariant measures for equation (1.1). Existence of invariant measures may be obtained from the results in [5]; indeed, the estimates to prove existence of a finite dimensional random attractor are similar to the ones used in [11] for existence of an invariant measure. But we prove it by another technique and with weaker assumptions. Moreover we do not restrict to the case of odd solutions. Finally, we tackle the problem of uniqueness of invariant measures.

We notice that the deterministic equation is a fourth-order PDE with the non linear term of the same form as in the Burgers or one-dimensional Navier-Stokes equations. We shall exploit this peculiarity in this paper, borrowing some techniques used in the analysis of the stochastic Burgers or Navier–Stokes equations.

This article is organized as follows. In Section 1 we introduce the stochastic Kuramoto–Sivashinsky equation as an Itô equation in Hilbert spaces. In Section 3 we prove a theorem of existence and uniqueness of solutions. In Section 4 we explain how to prove existence of invariant measures (the details are given in Section 5) and uniqueness (the details are given in Sections 6.1 and 6.2). Regularity results are proved in Section 6.3; in this way the results of irreducibility and strongly Feller property, proved first in the basic space H, are extended to more regular spaces. Section 7 presents the final theorem, covering all the results proved.

2 ABSTRACT SETTING

In this section, we introduce spaces and operators in order to define an abstract formulation of equation (1.1) as an Itô equation in Hilbert spaces.

SPACES

Let $\mathcal{P}$ be the space of periodic C^∞-functions defined on $[-\frac{L}{2}, \frac{L}{2}]$ and with zero mean. Closing this space with respect to the $L^2(-\frac{L}{2}, \frac{L}{2})$ and $H^2(-\frac{L}{2}, \frac{L}{2})$-norm we get the following spaces:

$$H = \{u \in L^2(-\tfrac{L}{2}, \tfrac{L}{2}) : \int_{-L/2}^{L/2} u(x)\ dx = 0\},$$

a Hilbert space with scalar product $\langle u, v\rangle_H = \int_{-L/2}^{L/2} u(x)v(x)\ dx$,

and

$$V = H \cap \{u \in H^2(-\tfrac{L}{2}, \tfrac{L}{2}) : u(-\tfrac{L}{2}) = u(\tfrac{L}{2}), u_x(-\tfrac{L}{2}) = u_x(\tfrac{L}{2})\},$$

a Hilbert space with scalar product $\langle u, v\rangle_V = \int_{-L/2}^{L/2} u_{xx}(x) v_{xx}(x)\, dx$.

OPERATORS

We define the operator A as

$$Au = -u_{xx}, \quad D(A) = V.$$

This is a linear operator in H, densely defined. It is strictly positive; its eigenvectors and eigenvalues are

$$\tilde{e}_{j,1}(x) = \sqrt{\tfrac{2}{L}} \sin(\tfrac{2j\pi}{L} x),\ \tilde{e}_{j,2}(x) = \sqrt{\tfrac{2}{L}} \cos(\tfrac{2j\pi}{L} x),$$

$$\tilde{\lambda}_j = \frac{4\pi^2}{L^2} j^2$$

for $j = 1, 2, \dots$. To shorten notations, from now on we shall denote by $\{e_j\}_{j=1}^{\infty}$ the sequence of the eigenvalues with corresponding eigenvectors λ_j (this is nothing but a relabelling of the sequence: $e_{2k-1} = \tilde{e}_{k,1}$, $e_{2k} = \tilde{e}_{k,2}$ and $\lambda_{2k-1} = \lambda_{2k} = \tilde{\lambda}_k$ for $k = 1, 2, \dots$). The sequence of the eigenvectors of A is a complete orthonormal basis of the space H. This implies that every $u \in H$ can be written as $u = \sum_j u_j e_j$, where the coefficients u_j satisfy the condition $\sum_j |u_j|^2 < \infty$.

The power operator A^α exists for any $\alpha \in \mathbb{R}$ (see, e.g., [12]), $D(A^\alpha) = \{u = \sum_j u_j e_j : \sum_j \lambda_j^{2\alpha} u_j^2 < \infty\}$, $A^\alpha u = \sum_j \lambda_j^\alpha u_j e_j$ and $|A^\alpha u|^2 = \sum_j \lambda_j^{2\alpha} u_j^2$. Given any α and β, the operator A^α is an isomorphism from $D(A^\beta)$ to $D(A^{\beta-\alpha})$. We have that $V = D(A)$ and the Poincaré inequality

$$|u|_V \geq \lambda_1 |u|_H.$$

Moreover, $D(A^2) = V \cap \{u \in H^4(-\tfrac{L}{2}, \tfrac{L}{2}) : u_{xx}(-\tfrac{L}{2}) = u_{xx}(\tfrac{L}{2}), u_{xxx}(-\tfrac{L}{2}) = u_{xxx}(\tfrac{L}{2})\}$ and $(D(A))' = D(A^{-1})$, where $(D(A))'$ is the dual space of $D(A)$ with respect to the duality of the H-scalar product. For any natural $m \in \mathbb{N}$, the $D(A^{\frac{m}{2}})$-norm is equivalent to the $H^m(-\tfrac{L}{2}, \tfrac{L}{2})$-norm.

In particular, for $\alpha > 0$, A^α is a self-adjoint operator in H generating a C_0-semigroup of linear operators in H:

$$e^{-A^\alpha t} u = \sum_j e^{-\lambda_j^\alpha t} u_j e_j \quad \text{for any } t \geq 0, u = \sum_j u_j e_j \in H.$$

In the following we shall use this result (see, e.g., [12]): for any $\beta > 0$ there exists a constant M_β such that

$$|A^{2\beta} e^{-A^2 t} u|_H \leq \frac{M_\beta}{t^\beta} |u|_H \quad \text{for any } t > 0, u \in H. \tag{2.1}$$

The bilinear operator B is studied investigating the associated trilinear form

$$b(u, v, z) = \langle B(u, v), z\rangle_H = \int_{-L/2}^{L/2} u(x) v_x(x) z(x)\, dx.$$

Using Hölder inequality and the continuous embedding of spaces $H^1(-\frac{L}{2},\frac{L}{2}) \subset L^\infty(-\frac{L}{2},\frac{L}{2})$, we obtain that there exists a constant c such that

$$|b(u_1,u_2,u_3)| \le |u_1|_H |(u_2)_x|_{L^\infty} |u_3|_H \le c|u_1|_H |Au_2|_H |u_3|_H \tag{2.2}$$

for $u_1, u_2, u_3 \in \mathcal{P}$. By density, the estimates hold for all $u_1, u_3 \in H$ and $u_2 \in D(A)$.

We shall use the following identities for elements of $\mathcal{P}$, obtained integrating by parts:

$$\left[\begin{array}{l} b(u,u,u) = 0, \\ b(u_1,u_2,u_2) = b(u_2,u_2,u_1) = -\frac{1}{2}b(u_2,u_1,u_2), \\ b(u_1,u_2,u_3) = -b(u_2,u_1,u_3) - b(u_1,u_3,u_2). \end{array}\right. \tag{2.3}$$

We collect here useful estimates. Hereafter the symbol c denotes different constants.

Proposition 2.1

$$|B(u,v)|_{D(A^{-1})} \le c|Au|_H |v|_H \tag{2.4}$$

$$|B(z,v)|_{D(A^{-1})} \le c|z|_H |Av|_H \tag{2.5}$$

$$|B(z,z)|_{D(A^{-1})} \le c|z|_H^2 \tag{2.6}$$

$$|B(u,v)|_{D(A^{-\delta})} \le c|u|_{D(A^{\frac{1}{2}-\delta})} |v|_{D(A^{\frac{1}{2}-\delta})} \qquad \textit{if } \ \delta \le 0 \tag{2.7}$$

Proof. First we show the inequality for elements of $\mathcal{P}$; then by density they hold true for elements in the spaces specified by the norms involved at each instance. We shall use repeatedly Hölder inequality and the fact that $H^1(-\frac{L}{2},\frac{L}{2})$ is continuously embedded in $L^\infty(-\frac{L}{2},\frac{L}{2})$, so that $|u|_{L^\infty} \le c|u_x|_H$. Moreover, $|B(u,v)|_{D(A^{-\alpha})} = \sup_{|w|_{D(A^\alpha)} \le 1} |b(u,v,w)|$.
For each inequality we show the main estimates to get it.

(2.4): we have $|B(u,v)|_{D(A^{-1})} \le c|A^{1/2}u|_H |v|_H$, since $|b(u,v,w)| \le |b(v,u,w)| + |b(u,w,v)|$ by (2.3) and $|b(v,u,w)| \le |v|_H |u_x|_H |w|_{L^\infty}$, $|b(u,w,v)| \le |u|_{L^\infty} |w_x|_H |v|_H$.

(2.5): $|B(z,v)|_{D(A^{-1})} \le \frac{1}{\lambda_1}|B(z,v)|_H \le \frac{1}{\lambda_1}|z|_H |v_x|_{L^\infty}$.

(2.6): $|b(z,z,w)| = \frac{1}{2}|b(z,w,z)|$ by (2.3) and $|b(z,w,z)| \le |z|_H^2 |w_x|_{L^\infty}$.

(2.7): we have $|uv_x|_H \le |u|_{L^\infty}|v_x|_H \le c|u|_{D(A^{1/2})}|v|_{D(A^{1/2})}$ and $|(uv_x)_x|_H \le |u_x v_x|_H + |uv_{xx}|_H \le |u_x|_H |v_x|_{L^\infty} + |u|_{L^\infty}|v_{xx}|_H \le c|u|_{D(A^{1/2})}|v|_{D(A)}$. Hence

$$|B(u,v)|_H \le c|u|_{D(A^{1/2})} |v|_{D(A^{1/2})}$$

and

$$|B(u,v)|_{D(A^{1/2})} \le c|u|_{D(A^{1/2})} |v|_{D(A)}.$$

By interpolation

$$|B(u,v)|_{D(A^\theta)} \le c|u|_{D(A^{1/2})} |v|_{D(A^{\frac{1}{2}+\theta})}, \qquad 0 < \theta < \frac{1}{2}.$$

We conclude the case of $-\frac{1}{2} \leq \delta < 0$ noting that $|u|_{D(A^{\frac{1}{2}})} \leq c_\delta |u|_{D(A^{\frac{1}{2}-\delta})}$.
For $\delta < -\frac{1}{2}$, set $\frac{m}{2} = -\delta$. $D(A^{\frac{m}{2}})$ is a multiplicative algebra for $m = 1, 2, \ldots$. Then the result is trivial for m integer; the estimate is even better:

$$|B(u,v)|_{D(A^{\frac{m}{2}})} = |uv_x|_{D(A^{\frac{m}{2}})} \leq c|u|_{D(A^{\frac{m}{2}})}|v|_{D(A^{\frac{1}{2}+\frac{m}{2}})}.$$

This allows to extend the result to any $m > 1$ as before by interpolation. □

To shorten notations, we write $B(u)$ for $B(u,u)$.

ABSTRACT FORMULATION

The abstract formulation of the initial value problem for equation (1.1) is

$$\begin{cases} du(t) + [\nu A^2 u(t) - Au(t) + B(u(t))]dt = G dw(t), \\ u(0) = y. \end{cases} \tag{2.8}$$

We have written the Wiener process as $Gw(t)$, where G is a linear operator and w is a cylindrical Wiener process in H defined on a probability space with filtration $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ (i.e. given a sequence $\{\beta_j\}_{j=1}$ of i.i.d. standard Wiener processes, we represent the Wiener process in series as $w(t) = \sum_j \beta_j(t) e_j$).

Hereafter, we denote by $|\cdot|$ the H-norm and by $\langle \cdot, \cdot \rangle$ the scalar product in H. If other norms are involved, they will be specified at each instance.
$\mathbb{E}$ denotes the expectation with respect to the probability measure $\mathbb{P}$.

3 SOLUTION: EXISTENCE, UNIQUENESS AND PROPERTIES

From now on, we assume that G is a linear operator in $\mathcal{P}$ such that

$$\sum_{j=1}^{\infty} \lambda_j^{-2} \sum_{k=1}^{\infty} [\langle Ge_k, e_j \rangle]^2 < \infty. \tag{3.1}$$

When possible, the operator G in defined as a linear operator in H (notice that $\mathcal{P}$ is dense in H) and this extension is again denoted by G. Then, condition (3.1) is equivalent to require $A^{-1}G$ be a Hilbert–Schmidt operator in H. We point out that [4] and [5] assume G be a Hilbert–Schmidt operator, which is stronger than (3.1).

Remark 3.1 *Keeping in mind the expression of the eigenvalues, we find that condition* (3.1) *is satisfied for instance if $G = LA^\gamma$ with $\gamma < \frac{3}{4}$ and L an isomorphism in H (we mean that L acts on the basis $\{e_j\}$ renaming it). Actually, for $0 < \gamma < \frac{3}{4}$, $G = LA^\gamma$ is a linear operator in H with domain $D(A^\gamma)$, whereas, for $\gamma \leq 0$, $G = LA^\gamma$ is a linear bounded operator in H. Usually the literature on stochastic equations in Hilbert spaces treats the case of G linear bounded operator in H; in this paper we assume $A^{-1}G$ be a linear bounded operator in H of Hilbert–Schmidt type.*

We point out that, in the physical literature, the noisy Kuramoto-Sivashinsky equation is presented as

$$\partial_t h(t,x) + \nu h_{xxxx}(t,x) + h_{xx}(t,x) - \tfrac{1}{2} h_x(t,x)^2 \ = \eta(t,x),$$

where ν is a positive surface diffusion coefficient and the variable h is the height profile of a surface eroded by ion sputtering. Setting $u = -h_x$ and $\partial_t w = -\eta_x$, we get (1.1) and the unknown u can be interpreted as a one-dimensional velocity field in a compressible fluid (see [3]).

[2] and [1] consider a centered Gaussian noise η with covariance

$$\mathbb{E}[\eta(t,x)\eta(t',x')] = \delta(t-t')\delta(x-x'),$$

i.e. $\eta(t,x) = \sum_j \dot\beta_j(t) e_j(x)$. This corresponds to have

$$Gw = \sum_{j \text{ even}} \beta_j \lambda_j^{1/2} e_{j-1} - \sum_{j \text{ odd}} \beta_j \lambda_j^{1/2} e_{j+1},$$

in our abstract equation (2.8), i.e. $G = LA^{1/2}$ where L is the linear bounded operator in H defined by $Le_j = (-1)^j e_{j+(-1)^{j+1}}$. Actually, L is an isomorphism in H. Then, (3.1) is satisfied.

On the other hand, [3] considers a noise with covariance

$$\mathbb{E}[\eta(t,x)\eta(t',x')] = \delta(t-t')[I+A]\delta(x-x'),$$

i.e. $\eta(t,x) = \sum_j \dot\beta_j(t)[I+A]^{1/2} e_j(x)$. This corresponds to have $G = LA^{1/2}[I+A]^{1/2}$ in our equation (2.8). In this case (3.1) is not satisfied.

We begin defining what is a solution for equation (2.8) in this work. Given an initial data $y \in H$, we consider solutions with less spatial regularity than whose of [4], [5].

Definition 3.2 *A stochastic process u is a weak solution of equation* (2.8) *on the time interval* $[0,T]$ *if,* $\mathbb{P}$*-a.s.*

$$u \in C([0,T];H),$$

it is progressively measurable and it satisfies the following identity

$$\langle u(t), h\rangle + \int_0^t \langle u(s), \nu A^2 h - Ah\rangle ds - \frac{1}{2}\int_0^t b(u(s), h, u(s)) ds = \langle y, h\rangle + \langle h, Gw(t)\rangle \tag{3.2}$$

for any $t \in [0,T]$ *and* $h \in \mathcal{P}$.

Relationship (3.2) is formally obtained from (2.8) multiplying by a test function h and integrating over the spatial domain; integration by parts as in (2.3) yields the above expression. Moreover, elementary calculus based on the Itô isometry gives

$$\mathbb{E}[\langle h, Gw(t)\rangle]^2 \leq \ t\,\Big(\sum_k \lambda_k^2 h_k^2\Big) \sum_j \lambda_j^{-2} \sum_{k=1}^{\infty} [\langle Ge_k, e_j\rangle]^2.$$

Bearing in mind (3.1) and (2.2), we have that all the terms in (3.2) are well defined.

Analyzing the equation for u, we first study the linear part. Notice that the linear operator $\nu A^2 - A$ is not strictly positive (this depends on L and ν, because its eigenvalues are $\nu\lambda_j^2 - \lambda_j$) and the negative eigenvalues cause instability for the linear Kuramoto–Sivashinsky equation.

Hence, we introduce the linear stochastic equation

$$dz_a(t) + \nu A^2 z_a(t)\ dt - Az_a(t)\ dt + az_a(t)\ dt = Gdw(t), \quad z_a(0) = \zeta. \tag{3.3}$$

We fixe a value $a > \frac{1}{4\nu}$ so to have $\nu\lambda_j^2 - \lambda_j + a > 0$ for all j. Therefore the operator $\nu A^2 - A + aI$, with domain $D(A^2)$, is strictly positive.
The process solving (3.3) is

$$z_a(t) = e^{-(\nu A^2 - A + a)t}\zeta + \int_0^t e^{-(\nu A^2 - A + a)(t-s)} Gdw(s).$$

Writing $e^{-(\nu A^2 - A + a)(t-s)} Gw(s) = \sum_{j,k} e^{-(\nu\lambda_j^2 - \lambda_j + a)(t-s)} \langle Ge_k, e_j\rangle \beta_k(s) e_j$, we have that

$$\mathbb{E}[z_a(t)] = e^{-(\nu A^2 - A + a)t}\zeta \in H,$$

$$\begin{aligned}\mathbb{E}\left[|z_a(t) - \mathbb{E}z_a(t)|^2\right] &= \sum_{j,k}\int_0^t e^{-2(\nu\lambda_j^2 - \lambda_j + a)(t-s)} |\langle Ge_k, e_j\rangle|^2 ds \\ &\leq \sum_{j,k} |\langle Ge_k, e_j\rangle|^2 \frac{1}{2(\nu\lambda_j^2 - \lambda_j + a)}.\end{aligned} \tag{3.4}$$

Therefore, if condition (3.1) holds, then for any time t, the Gaussian random variable $z_a(t) \in H$, $\mathbb{P}$-a.s. Using the factorization method as in [13], we get also that there exists a continuous version of z_a with values in H (and from now on we shall consider this continuous version).

To solve equation (2.8), we introduce a new unknown v_a as suggested in analysis of other Itô equations with additive noise (see, e.g., [11]). Set $v_a = u - z_a$ and $z_a(0) = 0$. Making the difference of the equations satisfied by u and z_a, we obtain that the equation satisfied by v_a does not contain the noise term. This is

$$\begin{cases} \frac{d}{dt} v_a(t) + \nu A^2 v_a(t) - Av_a(t) + B(v_a(t) + z_a(t)) = az_a(t), \\ v_a(0) = y. \end{cases} \tag{3.5}$$

We first prove a result for this problem, considered pathwise. This resembles the result for the deterministic equation (see [10]).

Proposition 3.3 *Assume* (3.1). *Then, for any $y \in H$ and $T > 0$ there exists a unique solution v_a for* (3.5) *such that*

$$v_a \in C([0,T]; H) \cap L^2(0,T; D(A))$$

$\mathbb{P}$-*a.s.*

Proof. We prove existence of a solution by means of the Galerkin method, i.e. we first deal with a finite dimensional problem for which there exists a solution and then we pass to the limit to recover the original evolutionary problem.

For any $n \in \mathbb{N}$, let Π_n be the orthogonal projector from H to the space spanned by $e_1, e_2, \dots, e_n$. Set $v_n = \Pi_n v$, $B_n = \Pi_n B$. Notice that the operators A and Π_n commute. The Galerkin system is

$$\begin{cases} \frac{d}{dt} v_a^n(t) + \nu A^2 v_a^n(t) - A v_a^n(t) + B_n(v_a^n(t) + z_a^n(t)) = a z_a^n(t), \\ v_a^n(0) = \Pi_n y. \end{cases} \tag{3.6}$$

We work pathwise. Since the coefficients are locally Lipschitz, there exists a unique solution, local in time. To show global existence we need a priori estimates. We take the scalar product of this equation with v_a^n, use (2.3), (2.2) and Young inequality, to obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|v_a^n|^2 + \nu |A v_a^n|^2 &= \langle A v_a^n, v_a^n\rangle + a\langle z_a^n, v_a^n\rangle - b(v_a^n, v_a^n, v_a^n) - b(v_a^n, z_a^n, v_a^n) \\ &\quad - b(z_a^n, v_a^n, v_a^n) - b(z_a^n, z_a^n, v_a^n) \\ &= \langle A v_a^n, v_a^n\rangle + a\langle z_a^n, v_a^n\rangle + b(z_a^n, v_a^n, v_a^n) + \tfrac{1}{2} b(z_a^n, v_a^n, z_a^n) \\ &\le |v_a^n||A v_a^n| + a|z_a^n||v_a^n| + c|z_a^n||v_a^n||A v_a^n| + \tfrac{1}{2} c |z_a^n|^2 |A v_a^n| \\ &\le \tfrac{1}{2}\nu |A v_a^n|^2 + c_\nu (1 + |z_a^n|^2)|v_a^n|^2 + \frac{a^2}{2}|z_a^n|^2 + c_\nu |z_a^n|^4. \end{aligned}$$

c_ν denotes different constants depending on ν and on the spatial domain. Then, there exists a constant C_1 (depending on ν and on the lenght L, but not on n) such that

$$\frac{d}{dt}|v_a^n|^2 + \nu |A v_a^n|^2 \le C_1(1 + |z_a|^2)|v_a^n|^2 + a^2 |z_a|^2 + C_1 |z_a|^4. \tag{3.7}$$

Applying Gronwall inequality to $\frac{d}{dt}|v_a^n|^2 \le C_1(1+|z_a|^2)|v_a^n|^2 + a^2|z_a|^2 + C_1|z_a|^4$ we get that

$$|v_a^n(t)|^2 \le |y|^2 e^{\int_0^t C_1(1+|z_a(s)|^2)ds} + \int_0^t e^{\int_s^t C_1(1+|z_a(\tau)|^2)d\tau}[a^2|z_a(s)|^2 + C_1|z_a(s)|^4]ds. \tag{3.8}$$

Since $\sup_{0\le s\le T} |z_a(s)|$ is finite $\mathbb{P}$-a.s. under assumption (3.1), we conclude that

$$\sup_{0\le t\le T} |v_a^n(t)|^2 \le C_2,$$

where C_2 is independent of n. Now, integrating in time (3.7), by means of the last estimate we get also that

$$\int_0^T |A v_a^n(t)|^2 dt \le C_3,$$

where C_3 is independent of n. Hence, $v_a^n \in C([0,T]; H) \cap L^2(0,T; D(A))$.

Finally, we note that

$$\frac{d}{dt}v_a^n = -\nu A^2 v_a^n + A v_a^n + a z_a^n - B_n(v_a^n) - B_n(z_a^n) - B_n(v_a^n, z_a^n) - B_n(z_a^n, v_a^n).$$

The r.h.s. belongs to $L^2(0,T;D(A^{-1}))$; this is easy to check for the first three terms, whereas for the terms involving B_n we have to bring to mind (2.4)-(2.6).

Summing up, the Galerkin sequence v_a^n is bounded in $L^\infty(0,T;H) \cap L^2(0,T;D(A)) \cap H^1(0,T;D(A^{-1}))$. Since the space $L^2(0,T;D(A)) \cap H^1(0,T;D(A^{-1}))$ is compactly embedded in $L^2(0,T;H)$ (see, e.g., [14] at pg. 271), we conclude that there exists a subsequence $v_a^{n_k}$ and a limit v_a such that

$v_a^{n_k}$ converges to v_a weakly in $L^2(0,T;D(A))$,
$v_a^{n_k}$ converges to v_a $\star$-weakly in $L^\infty(0,T;H)$,
$v_a^{n_k}$ converges to v_a strongly in $L^2(0,T;H)$.

These convergences grant that v_a is a solution of equation (3.5); notice that the strong convergence allows to pass to the limit in the non linear term B (see details in [14], dealing with the Navier–Stokes equation which has this same non linearity).

Finally, if $v_a \in L^2(0,T;D(A))$, $v_a' \in L^2(0,T;D(A^{-1}))$, then $v_a \in C([0,T];H)$ (see [14], Chapter III Lemma 1.2).

Uniqueness is easy to check. We shall prove it for u in the next theorem and the method applies also to v_a (usually it is more difficult to get uniqueness for u, because u is less regular than v_a; for this reason we give details only for u). $\square$

The result for the process u is the following.

Theorem 3.4 *Assume* (3.1). *Then, for any $y \in H$ and $T > 0$ there exists a unique solution u for* (2.8) *as defined in Definition 3.2 such that*

$$u \in C([0,T];H)$$

$\mathbb{P}$-a.s.. Moreover, u is a Markov process in H, which is Feller in H.

Proof. The process $u = v_a + z_a$ is a solution of (2.8) by construction and the regularity of v_a and z_a provides $u \in C([0,T];H)$, $\mathbb{P}$-a.s.

As far as we are concerned with the the uniqueness, let $u_1, u_2 \in C([0,T];H)$ be two solutions of equation (2.8) with the same initial data. Set $U = u_1 - u_2$. Then this difference satisfies the following equation

$$\frac{dU}{dt} + \nu A^2 U - AU + B(u_1) - B(u_2) = 0; \qquad U(0) = 0.$$

By the bilinearity of B: $B(u_1) - B(u_2) = B(U, u_1) + B(u_2, U)$. Then

$$\frac{dU}{dt} + \nu A^2 U = AU - B(U, u_1) - B(u_2, U).$$

Taking the scalar product in H of this equation with U and proceeding to estimate the terms as done before for v_a, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|U|^2 + \nu|AU|^2 &\leq |U||AU| + c|U||AU|(|u_1| + |u_2|) \\ &\leq \frac{\nu}{2}|AU|^2 + c_\nu(1 + |u_1|^2 + |u_2|^2)|U|^2.\end{aligned}$$

Actually, we should work first with the finite dimensional approximation and then pass to the limit, obtaining that $U \in C([0,T];H) \cap L^2(0,T;D(A))$. The fact that U is more regular than u_1 and u_2 justifies all the estimates.

From (3.9) we conclude by Gronwall lemma that

$$\sup_{t\in[0,T]} |U(t)|^2 \leq |U(0)|^2 e^{2c_\nu \int_0^T (1+|u_1(s)|^2+|u_2(s)|^2)ds}. \tag{3.9}$$

Then

$$|U(t)| = 0 \text{ for all } t \in [0,T]$$

and uniqueness is proved.

Since the processes v_a^n and z_a^n are progressively measurable, so are also u^n and in the limit u too. The same for the Markov property, i.e. given $0 < t_1 < \cdots < t_m \leq T$

$$\mathbb{P}\{u(t_m) \in \Gamma | u(t_1), \dots, u(t_{m-1})\} = \mathbb{P}\{u(t_m) \in \Gamma | u(t_{m-1})\}$$

for any Borelian subset Γ of H.

Finally, denoting by $u(\cdot; y)$ the solution to equation (2.8) with initial data $y \in H$, the Markovian transition semigroup $\{P_t\}_{t\geq 0}$, defined by

$$(P_t\phi)(y) = \mathbb{E}\phi(u(t;y)),$$

is well defined on the space of Borelian bounded functions $B_b(H)$. Moreover, it is Feller, that is $P_t : C_b(H) \to C_b(H)$. Indeed, from (3.9), we get that the solution u depends continuously on the initial data; then for any t

$$\lim_{y_1\to y_2} |u(t;y_1) - u(t;y_2)| = 0 \quad \mathbb{P} - a.s..$$

Given any continuous function $\phi : H \to \mathbb{R}$, we have that $\phi(u(t;y_1)) \to \phi(u(t;y_2))$ pathwise as $y_1 \to y_2$. Since ϕ is bounded, by dominated convergence theorem we conclude that $\mathbb{E}\phi(u(t;y_1)) \to \mathbb{E}\phi(u(t;y_2))$ as $y_1 \to y_2$. □

If the process z_a, solution to the linear stochastic equation (3.3), is more regular, then also $u = v_a + z_a$ is more regular. We note that, by interpolation,

$$|A^\alpha v| \leq c|v|^{1-\alpha}|Av|^\alpha$$

for $0 < \alpha < 1$. Hence, if $v_a \in C([0,T];H) \cap L^2(0,T;D(A))$, we have $v_a \in L^{2/\alpha}(0,T;D(A^\alpha))$.

Moreover if

$$\sum_{j=1}^{\infty} \lambda_j^{2(\alpha-1)} \sum_{k=1}^{\infty} [\langle Ge_k, e_j \rangle]^2 < \infty \tag{3.10}$$

then $z_a \in C([0,T]; D(A^\alpha))$ $\mathbb{P}$-a.s. . Indeed the estimates to prove this are easily obtained as in (3.4). We write them for a general initial data $\zeta \in D(A^\alpha)$:

$$\begin{aligned} \mathbb{E}[z_a(t)] &= e^{-(\nu A^2 - A + a)t}\zeta \in D(A^\alpha), \\ \mathbb{E}\left[|z_a(t) - \mathbb{E}z_a(t)|^2_{D(A^\alpha)} \right] &\leq \sum_{j,k} |\langle Ge_k, e_j \rangle|^2 \frac{\lambda_j^{2\alpha}}{2(\nu\lambda_j^2 - \lambda_j + a)}. \end{aligned} \tag{3.11}$$

This implies the following result, which enforces the previous theorem.

Corollary 3.5 *(i) If* (3.10) *holds for some* $0 < \alpha < 1$*, then there exists a unique process u solution to* (2.8)*, which in addition to the properties of Theorem 3.4 has* $u \in C([0,T]; H) \cap L^{2/\alpha}(0,T; D(A^\alpha))$ $\mathbb{P}$*-a.s.*

(ii) If (3.10) *holds for some* $\alpha \geq 1$*, then there exists a unique process u solution to* (2.8)*, which in addition to the properties of Theorem 3.4 has* $u \in C([0,T]; H) \cap L^2(0,T; D(A))$ $\mathbb{P}$*-a.s.*

4 INVARIANT MEASURES

Definition 4.1 *A probability measure* μ *on* H *is invariant for the Markovian semigroup* $\{P_t\}_{t\geq 0}$ *associated to equation* (2.8) *if*

$$\int P_t \phi d\mu = \int \phi d\mu \qquad \textit{for all } \phi \in C_b(H), \ t \geq 0.$$

Introducing the semigroup $\{P_t^*\}_{t\geq 0}$, acting on probability measures on H, as $\int \phi \, dP_t^* \mu = \int P_t \phi \, d\mu \equiv < P_t\phi, \mu >$, a measure μ is invariant if

$$P_t^* \mu = \mu \text{ for all } t \geq 0,$$

that is, μ is a fixed point for the evolution of probability measures under P_t^*.

Since the Markovian semigroup $\{P_t\}_{t\geq 0}$ is Feller in H, we use the well-known Krylov–Bogoliubov method to prove existence of invariant measures. Namely, if the family of measures

$$\mu_T = \frac{1}{T} \int_1^T P_s^* \delta_0 ds, \qquad T > 1,$$

(where $P_s^* \delta_0$ is the law of the process $u(s;0)$) is tight in H, then there exists a subsequence μ_{T_k} weakly convergent to a measure μ, as $k \to \infty$ (and $T_k \to \infty$). Then this limit measure μ is invariant for the Markovian semigroup $\{P_t\}_{t\geq 0}$;

indeed, for any $\phi \in C_b(H)$

$$\begin{aligned} < P_t\phi, \mu > &= \lim_{k\to\infty} \tfrac{1}{T_k} \int_1^{T_k} < P_t\phi, P_s^*\delta_0 > ds \\ &= \lim_{k\to\infty} \tfrac{1}{T_k} \int_1^{T_k} < \phi, P_{t+s}^*\delta_0 > ds \\ &= \lim_{k\to\infty} \tfrac{1}{T_k} \int_{t+1}^{t+T_k} < \phi, P_s^*\delta_0 > ds \\ &= \lim_{k\to\infty} \tfrac{1}{T_k} \int_1^{T_k} < \phi, P_s^*\delta_0 > ds \\ &\quad + \lim_{k\to\infty} \tfrac{1}{T_k} [\int_{T_k}^{t+T_k} < \phi, P_s^*\delta_0 > ds - \int_1^{t+1} < \phi, P_s^*\delta_0 > ds] \\ &=< \phi, \mu > . \end{aligned}$$

As far as we are concerned with uniqueness of invariant measures, we prove it along the lines of [15]. Let $P(t, y, \cdot)$ denote the transition probability: $P(t, y, \Gamma) = \mathbb{P}\{u(t; y) \in \Gamma\}$. By Khas'minskii theorem, if a Markovian semigroup $\{P_t\}_{t\geq 0}$ is irreducible at time $t_1 > 0$ and strongly Feller at time $t_2 > 0$, then it is regular at time $t_1 + t_2$, that is the transition probabilities $P(t, y, \cdot)$ are equivalent for $t > t_1 + t_2, y \in H$. By Doob theorem, given an invariant measure μ, if the Markovian semigroup $\{P_t\}_{t\geq 0}$ is regular for some $t_0 > 0$, then μ is strongly mixing and

$$\lim_{t\to\infty} P(t, y, \Gamma) = \mu(\Gamma)$$

for arbitrary $y \in H$ and Borelian subset Γ of H.
Moreover, μ is the unique invariant measure, is ergodic and is equivalent to any transition probability measure $P(t, y, \cdot)$ for $y \in H$ and $t \geq t_0$.

Since the irreducibility and strongly Feller property are interesting in themselves for a stochastic equation, we prove them separately in Section 6.1 and 6.2, respectively. First, we prove them in the space H; then, by regularity results in Section 6.3 we prove them in more regular spaces.

5 EXISTENCE OF INVARIANT MEASURES

In this section, we work with an operator G satisfying (3.10) for some $\alpha > 0$.
Keeping in mind that the space $D(A^{\tilde\alpha})$ is compactly embedded in H for any $\tilde\alpha > 0$, our aim is to show that there exists a parameter $\tilde\alpha > 0$ for which the following holds true:

$$\forall\varepsilon > 0 \quad \exists R > 0 : \ \frac{1}{T}\int_1^T \mathbb{P}\{|A^{\tilde\alpha}u(t; 0)| > R\}dt < \varepsilon \text{ for all } T > 1. \tag{5.1}$$

Notice that we consider initial data equal to zero and therefore the regularity of the solution u depends only on G.

First of all, the linear equation enjoys property (5.1). Indeed, $z_a(t) = \int_0^t e^{-(\nu A^2 - A + a)(t-s)} G dw(s)$ and from (3.11) we know that $\sup\limits_{0\leq t<\infty} \mathbb{E}|A^\alpha z_a(t)|^2$

is finite and tends to 0 as $a \to \infty$. Moreover, by Chebyshev inequality

$$\begin{aligned}\frac{1}{T}\int_0^T \mathbb{P}\{|A^\alpha z_a(t)| > R\}dt &\le \frac{1}{T}\int_0^T \frac{\mathbb{E}(|A^\alpha z_a(t)|^2)}{R^2}dt \\ &\le \frac{1}{R^2}\sup_{0\le t<\infty}\mathbb{E}(|A^\alpha z_a(t)|^2)\end{aligned} \tag{5.2}$$

and we conclude that $\frac{1}{T}\int_0^T \mathbb{P}\{|A^\alpha z_a(t)| > R\}dt$ can be made as small as we want by a suitable choice of R.

Now, we prove tightness for u, looking the equation satisfied by u as a perturbation of the linear equation (3.3). We follow [16] (see also [15]). As a first step, let us prove

Proposition 5.1 *Assume* (3.1). *Then*

$$\forall\varepsilon > 0 \quad \exists R > 0 : \ \frac{1}{T}\int_0^T \mathbb{P}\{|u(t;0)| > R\}dt < \varepsilon \quad \textit{for all } T > 0. \tag{5.3}$$

Proof. By (5.2), this proposition holds true if we prove (5.3) for $v_a = u - z_a$.

First, a priori estimates are required and we borrow from the deterministic case the suitable bounds. The unknown v_a satisfies (3.5) with $v_a(0) = 0$. We have proved that there exists a unique process solution such that $v_a \in C([0,T];H) \cap L^2(0,T;D(A))$ $\mathbb{P}$-a.s. and

$$\frac{d}{dt}|v_a|^2 + \nu|Av_a|^2 \le C_1(1 + |z_a|^2)|v_a|^2 + a^2|z_a|^2 + C_1|z_a|^4. \tag{5.4}$$

Applying Gronwall lemma, we do not find useful estimates to prove (5.3). Hence, we proceed as in the deterministic case. We follow [17] (a similar result is in [18]) and introduce an auxiliary function h_s, where $h_s(x) = h(x+s)$ for a suitable $h \in D(A)$ and $s = s(t)$; other properties of h will be presented below. We work with $v_a - h_s \equiv v_a(t,x) - h(x + s(t))$; we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|v_a - h_s|^2 &= \langle \frac{d}{dt}v_a - \frac{d}{dt}h_s, v_a - h_s\rangle \\ &= -\nu\langle A^2v_a, v_a\rangle + \langle Av_a, v_a\rangle - \langle B(v_a), v_a - h_s\rangle + \nu\langle Av_a, Ah_s\rangle \\ &\quad - \langle Av_a, h_s\rangle + a\langle z_a, v_a - h_s\rangle - \langle B(v_a, z_a) + B(z_a, v_a), v_a - h_s\rangle \\ &\quad - \langle B(z_a), v_a - h_s\rangle - \langle h_s'\dot{s}, v_a - h_s\rangle \\ &\le -\nu\langle A^2v_a, v_a\rangle + \langle Av_a, v_a\rangle + \langle B(v_a), h_s\rangle + \nu\langle Av_a, Ah_s\rangle \\ &\quad - \langle Av_a, h_s\rangle + a\langle z_a, v_a - h_s\rangle - \langle B(v_a, z_a) + B(z_a, v_a), v_a - h_s\rangle \\ &\quad - \langle B(z_a), v_a - h_s\rangle,\end{aligned}$$

where we used (see [17]) that $\langle h_s'\dot{s}, v_a\rangle = c\dot{s}^2 \ge 0$ and $\langle h_s', h_s\rangle = 0$.
Generalizing the proof of [17] (indeed, they consider $\nu = 1$), we have that for any $\nu > 0$ there exists a positive constant κ, depending on L and ν, such that

$$-\nu\langle A^2v_a, v_a\rangle + \langle Av_a, v_a\rangle + \langle B(v_a), h_s\rangle \le -\kappa\langle A^2v_a, v_a\rangle - \kappa|v_a|^2.$$

Therefore, using (2.3), (2.2) and Young inequality

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}|v_a - h_s|^2 + &\kappa\langle A^2 v_a, v_a\rangle + \kappa|v_a|^2 \\
&\leq \nu\langle Av_a, Ah_s\rangle - \langle Av_a, h_s\rangle + a\langle z_a, v_a - h_s\rangle \\
&\quad + 2b(z_a, v_a, v_a - h_s) + b(v_a, v_a - h_s, z_a) + \frac{1}{2}b(z_a, v_a - h_s, z_a) \\
&\leq \nu|Av_a||Ah_s| + |Av_a||h_s| + a|z_a||v_a - h_s| \\
&\quad + c|z_a||Av_a||v_a - h_s| + c|v_a||A(v_a - h_s)||z_a| + c|z_a|^2|A(v_a - h_s)| \\
&\leq \frac{\kappa}{3}|Av_a|^2 + \frac{\kappa}{3}|A(v_a - h_s)|^2 + c_\nu|z_a|^2|v_a - h_s|^2 + c_\nu|z_a|^2|v_a|^2 \\
&\quad + a|z_a||v_a - h_s| + c_\nu|z_a|^4 + c_\nu|Ah_s|^2 + c_\nu|h_s|^2
\end{aligned}
$$

By means of the triangle inequality and of embedding $D(A) \subset H$, we get that there exist positive constants κ and $\bar{c}$ (depending on L and ν) such that

$$
\frac{d}{dt}|v_a - h_s|^2 + \kappa|v_a - h_s|^2 \leq \bar{c}|z_a|^2|v_a - h_s|^2 + \bar{c}(1 + |Ah|^4 + |z_a|^4) \tag{5.5}
$$

since $|h_s| = |h|$.
From [16], we define $\sigma(t) = \log(|v_a(t) - h_{s(t)}|^2 \vee R)$ for some $R > 1$ to be chosen later. Denoting by χ_Γ the indicator function of the set Γ, we have

$$
\int_0^T \chi_{\{|v_a(t) - h_{s(t)}|^2 \geq R\}} \frac{\frac{d}{dt}|v_a(t) - h_{s(t)}|^2}{|v_a(t) - h_{s(t)}|^2} dt = \sigma(T) - \sigma(0) \geq 0. \tag{5.6}
$$

Multiplying (5.5) by $\chi_{\{|v_a(t)-h|^2 \geq R\}} \frac{1}{|v_a(t)-h|^2}$, we obtain

$$
\begin{aligned}
\chi_{\{|v_a(t) - h_{s(t)}|^2 \geq R\}} &\frac{\frac{d}{dt}|v_a(t) - h_{s(t)}|^2}{|v_a(t) - h_{s(t)}|^2} + \kappa\chi_{\{|v_a(t) - h_{s(t)}|^2 \geq R\}} \\
&\leq \bar{c}|z_a(t)|^2 + \frac{\bar{c}}{R}(1 + |Ah|^4 + |z_a(t)|^4).
\end{aligned}
$$

Integrating in time, bearing in mind (5.6) and taking expectation we get

$$
\begin{aligned}
\frac{\kappa}{T}\int_0^T &\mathbb{P}\{|v_a(t) - h_{s(t)}| \geq R\} dt \\
&\leq \frac{\bar{c}}{R}(1 + |Ah|^4) + \frac{\bar{c}}{R}\frac{1}{T}\int_0^T \mathbb{E}|z_a(t)|^4 dt + \bar{c}\sup_{0 \leq t < \infty} \mathbb{E}|z_a(t)|^2.
\end{aligned}
$$

Taking a sufficiently large the last term can be as small as we want; moreover, for this fixed a the two other terms in the r.h.s. tends to zero as R tends to ∞. On the other hand, by triangle inequality we find that the same estimate holds for the quantity $\frac{1}{T}\int_0^T \mathbb{P}\{|v_a(t)| \geq R\} dt$. Thus, the proposition is proved. $\square$

Now we prove (5.1).

Proposition 5.2 *Let* (3.10) *be satisfied for some* $\alpha > 0$. *Then there exists* $\tilde{\alpha} \in (0,1)$ *such that*

$$\forall \varepsilon > 0 \quad \exists R > 0 : \ \frac{1}{T}\int_1^T \mathbb{P}\{|A^{\tilde{\alpha}}u(t;0)| > R\}dt < \varepsilon \text{ for all } T \geq 1. \tag{5.7}$$

Moreover, there exists at least one invariant measure for equation (2.8)*.*

Proof. Since $A^\alpha u = A^\alpha v_a + A^\alpha z_a$, according to (5.2) we seek the result for $A^\alpha v_a$. To deal with $A^\alpha v_a$, we exploit the regularizing effect of the semigroup $e^{-\nu A^2 t}$. Let us write equation (3.5), on the time interval $[t, t+1]$, in the integral form:

$$v_a(t+1) = e^{-\nu A^2}v_a(t) + \int_t^{t+1} e^{-\nu A^2(t+1-s)}Av_a(s)\ ds \\ - \int_t^{t+1} e^{-\nu A^2(t+1-s)}B(u(s))\ ds + a\int_t^{t+1} e^{-\nu A^2(t+1-s)}z_a(s)\ ds.$$

Then

$$A^\alpha v_a(t+1) = A^\alpha e^{-\nu A^2}v_a(t) + \int_t^{t+1} A^{\alpha+1}e^{-\nu A^2(t+1-s)}v_a(s)\ ds \\ - \int_t^{t+1} A^{\alpha+1}e^{-\nu A^2(t+1-s)}A^{-1}B(u(s))\ ds + a\int_t^{t+1} e^{-\nu A^2(t+1-s)}A^\alpha z_a(s)\ ds. \tag{5.8}$$

By means of (2.1), we estimate each term in the r.h.s.

$$|A^\alpha e^{-\nu A^2}v_a(t)| \leq K_1|v_a(t)|,$$

$$\left|\int_t^{t+1} A^{\alpha+1}e^{-\nu A^2(t+1-s)}v_a(s)\ ds\right| \leq \ \tilde{K}_2 \sup_{0\leq r\leq 1}|v_a(t+r)|,$$

where $\tilde{K}_2 := \displaystyle\int_t^{t+1} \frac{M_{\frac{\alpha+1}{2}}}{(t+1-s)^{\frac{\alpha+1}{2}}}ds$ is finite if $\alpha < 1$. If $\alpha \geq 1$, we choose $\tilde{\alpha}$ such that $0 < \tilde{\alpha} < 1 \leq \alpha$ and obtain this result for $\tilde{\alpha}$ instead of α.

$$\begin{aligned}
&\left|\int_t^{t+1} A^{\alpha+1}e^{-\nu A^2(t+1-s)}A^{-1}B(u(s))\ ds\right| \\
&\quad\leq \int_t^{t+1} \frac{M_{\frac{\alpha+1}{2}}}{(t+1-s)^{\frac{\alpha+1}{2}}}|B(u(s))|^2_{D(A^{-1})}ds \\
&\quad\overset{\text{by (2.6)}}{\leq} \int_t^{t+1} \frac{M_{\frac{\alpha+1}{2}}}{(t+1-s)^{\frac{\alpha+1}{2}}}c|u(s)|^2 ds \\
&\quad\leq 2c\sup_{0\leq r\leq 1}|v_a(t+r)|^2\tilde{K}_2 + 2cM_{\frac{\alpha+1}{2}}\int_t^{t+1}\frac{|z_a(s)|^2}{(t+1-s)^{\frac{\alpha+1}{2}}}ds \\
&\quad\leq 2c\sup_{0\leq r\leq 1}|v_a(t+r)|^2\tilde{K}_2 + 2cM_{\frac{\alpha+1}{2}}\tilde{K}_3\Big(\int_t^{t+1}|z_a(s)|^p ds\Big)^{2/p},
\end{aligned}$$

where $\tilde{K}_3 := \Big(\int_t^{t+1} \frac{ds}{(t+1-s)^{\frac{\alpha+1}{2}\frac{p}{p-2}}} \Big)^{\frac{p-2}{p}}$ is finite for suitable $p > 2$ depending on $\alpha \in (0,1)$. Again, if $\alpha \geq 1$, we obtain the result for $\tilde{\alpha}$ such that $0 < \tilde{\alpha} < 1 \leq \alpha$. The last bound is

$$
\begin{aligned}
|\int_t^{t+1} e^{-\nu A^2(t+1-s)} A^\alpha z_a(s)\ ds| &\leq \int_t^{t+1} |A^\alpha z_a(s)|\ ds \\
&\leq \frac{1}{2}\Big(\int_t^{t+1} |A^\alpha z_a(s)|^p ds \Big)^{2/p} + \frac{1}{2}.
\end{aligned}
$$

Coming back to (5.8), these bounds imply that there exists $\tilde{\alpha} \in (0,1)$ such that

$$
|A^{\tilde{\alpha}} v_a(t+1)| \leq K_1 |v_a(t)| + K_2 \sup_{0 \leq r \leq 1} |v_a(t+r)|^2 + K_3 \Big(\int_t^{t+1} |A^\alpha z_a(s)|^p ds \Big)^{2/p} + K_4
$$

for suitable constants K_1, K_2, K_3, K_4. Then

$$
\begin{aligned}
\mathbb{P}\{|A^{\tilde{\alpha}} v_a(t+1)| > R + K_4\} \leq\ & \mathbb{P}\{K_1 |v_a(t)| > \frac{R}{3}\} \\
&+ \mathbb{P}\{K_2 \sup_{0 \leq r \leq 1} |v_a(t+r)|^2 > \frac{R}{3}\} \\
&+ \mathbb{P}\{K_3 \Big(\int_t^{t+1} |A^\alpha z_a(s)|^p ds \Big)^{2/p} > \frac{R}{3}\}.
\end{aligned}
$$

Bearing in mind the proof of Proposition 5.1, we deal with the first term in the r.h.s. in order to make it as small as we want by a suitable choice of R. The same holds for the third term, using Chebyshev inequality and the fact that, for $p/2$ integer, $\mathbb{E}|A^\alpha z_a(s)|^p = c_p [\mathbb{E}|A^\alpha z_a(s)|^2]^{p/2}$ by Gaussianity. We are left with the second term to analyze. We apply Gronwall lemma to inequality (5.4) and get

$$
\begin{aligned}
\sup_{0 \leq r \leq 1} |v_a(t+r)|^2 \leq\ & |v_a(t)|^2 e^{\int_t^{t+1} C_1(1+|z_a(s)|^2) ds} \\
&+ \int_t^{t+1} e^{\int_s^{t+1} C_1(1+|z_a(r)|^2) dr} (a^2 |z_a(s)|^2 + C_1 |z_a(s)|^4) ds.
\end{aligned}
$$

Considering the probabilities, we have

$$
\begin{aligned}
\mathbb{P}\{\sup_{0 \leq r \leq 1} |v_a(t+r)|^2 > R_2\} &\leq \mathbb{P}\{|v_a(t)|^2 e^{\int_t^{t+1} C_1(1+|z_a(s)|^2) ds} > \frac{R_2}{2}\} \\
&\quad + \mathbb{P}\{\int_t^{t+1} e^{\int_s^{t+1} C_1(1+|z_a(r)|^2) dr} (a^2 |z_a(s)|^2 + C_1 |z_a(s)|^4) ds > \frac{R_2}{2}\} \\
&\leq \mathbb{P}\{|v_a(t)|^2 > \sqrt{\frac{R_2}{2}}\} + 2\mathbb{P}\{e^{\int_t^{t+1} C_1(1+|z_a(s)|^2) ds} > \sqrt{\frac{R_2}{2}}\} \\
&\quad + \mathbb{P}\{a^2 |z_a(s)|^2 + C_1 |z_a(s)|^4 > \sqrt{\frac{R_2}{2}}\}.
\end{aligned}
$$

Once more, we deal with the first term in the last r.h.s according to Proposition 5.1 and with the two other terms by means of Chebyshev inequality.
This concludes the proof of (5.7). The existence of invariant measures follows from Krylov–Bogoliubov method. □

6 UNIQUENESS OF INVARIANT MEASURES

We first prove irreducibility and strongly Feller property in the space H. Then in Section 6.3, we extend these results to more regular spaces $D(A^\alpha)$, $\alpha > 0$, by means of Girsanov theorem.

6.1 Irreducibility

We say that the Markovian semigroup $\{P_t\}_{t\geq 0}$ is H-irreducible at time $t > 0$ if, for arbitrary non empty open set $\Gamma \subseteq H$ and all $y \in H$,

$$P_t \chi_\Gamma(y) > 0.$$

This is equivalent to

$$\mathbb{P}\{|u(t;y) - \tilde{u}| < R\} > 0 \qquad \forall t > 0 \quad \forall y, \tilde{u} \in H \quad \forall R > 0.$$

Given any $t > 0$, we prove it following an idea from [20]: we show pathwise that there are suitable $\bar{R} > 0$ and $\bar{z}_a \in C_0([0,T];H)$ (the subset of $C([0,T];H)$-functions which vanish at $t = 0$) such that

$$\{|u(t;y) - \tilde{u}| < R\} \supseteq \{|z_a - \bar{z}_a|_{C_0([0,t];H)} < \bar{R}\} \tag{6.1}$$

and that the probability of the last set is strictly positive.

By [15] (see also [19]), if G is a linear bounded operator in H with range dense in H, then the law of the process $z_a(\cdot;0)$ is full in $C_0([0,T];H)$, i.e.

$$\mathbb{P}\{z_a \in \Lambda\} > 0 \qquad \text{for any non empty open set } \Lambda \subset C_0([0,T];H).$$

Notice that this covers the case of $G = A^\gamma$ for $\gamma \leq 0$. But Lemma 2.6 in [19] works also for $0 < \gamma < \frac{3}{4}$; the hypothesis is checked with calculations similar to (3.11).

Hence, what remains to prove is (6.1). First, we define a function $\bar{u}$ linking y to $\tilde{u}$ in time t as

$$\bar{u}(s) = y + \frac{s}{t}[\tilde{u} - y] \qquad \text{for } 0 \leq s \leq t.$$

We have that $\bar{u} \in C([0,t];H)$. Therefore

$$\{|u(t;y) - \tilde{u}| < R\} \supseteq \{|u(\cdot;y) - \bar{u}(\cdot;y)|_{C([0,t];H)} < R\}. \tag{6.2}$$

We work now with the $C([0,t];H)$-norms and prove that given $\bar u \in C([0,t];H)$ there exist $\bar v_a, \bar z_a \in C([0,t];H)$ such that $\bar u = \bar v_a + \bar z_a$, $\bar v_a$ satisfies equation (3.5) with $z_a = \bar z_a$ and $\bar z_a(0) = 0$; moreover

$$|v_a(\cdot;y) - \bar v_a(\cdot;y)|_{C([0,t];H)} \leq L|z_a - \bar z_a|_{C_0([0,t];H)}. \tag{6.3}$$

Indeed, equation (3.5) satisfied by v_a can be written as

$$\begin{cases} \frac{d}{dt}\bar v_a(t) + [\nu A^2 - A + a]\bar v_a(t) = au(t) - B(\bar u(t)) \\ \bar v_a(0) = y \end{cases}$$

By means of the estimates used in the previous sections, it is easy to check[1] that, given $\bar u \in C([0,t];H)$, the r.h.s. $a\bar u - B(\bar u) \in C([0,t];D(A^{-1}))$. Then, by classical results on linear parabolic equations, we know that there exists a unique solution $\bar v_a \in C([0,t];H) \cap L^2(0,t;D(A))$.
Therefore the function $\bar z_a := \bar u - \bar v_a$ is well defined and belongs to $C_0([0,t];H)$.

The difference $V_a := v_a - \bar v_a$ satisfies the equation

$$\begin{cases} \frac{d}{dt}V_a + \nu A^2 V_a & -AV_a + B(V_a, v_a) + B(\bar v_a, V_a) + B(V_a, z_a) + B(\bar v_a, Z_a) \\ & +B(Z_a, v_a) + B(\bar z_a, V_a) = AZ_a - B(Z_a, z_a) - B(\bar z_a, Z_a) \\ V_a(0) = 0 & \end{cases}$$

(where $Z_a = z_a - \bar z_a$).

Taking the scalar product in H of the first equation with V_a and using estimates on the trilinear form as usual, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|V_a|^2 + \nu|AV_a|^2 &\leq c(|v_a| + |\bar v_a| + |z_a| + |\bar z_a|)|V_a||AV_a| + c|A\bar v_a||V_a||Z_a| \\ &\quad + c(1 + |\bar v_a| + |z_a| + |\bar z_a|)|Z_a||AV_a| \\ &\leq \frac{\nu}{2}|AV_a|^2 + \psi_1|V_a|^2 + \psi_2|Z_a|^2 \end{aligned}$$

where $\psi_1 = c_\nu(1 + |v_a|^2 + |\bar v_a|^2 + |z_a|^2 + |\bar z_a|^2) \in C([0,t];H)$ and $\psi_2 = c_\nu(1 + |A\bar v_a|^2 + |z_a|^2 + |\bar z_a|^2) \in L^1(0,t;H)$.

By Gronwall lemma, we conclude that

$$\sup_{0\leq s\leq t} |V_a(s)|^2 \leq \int_0^t e^{\int_s^t \psi_1(r)dr}\psi_2(s)\,|Z_a(s)|^2 ds.$$

Hence

$$\sup_{0\leq s\leq t} |V_a(s)|^2 \leq L \sup_{0\leq s\leq t} |Z_a(s)|^2$$

for some constant L depending on t, on the H- norm of y and the $C([0,t];H)$-norm of z_a, $\bar z_a$. This is (6.3); it implies that

$$|u(\cdot;y) - \bar u(\cdot;y)|_{C([0,t];H)} \leq (1+L)|z_a - \bar z_a|_{C_0([0,t];H)}$$

and by (6.2), the relationship (6.1) follows with $R = (1+L)\bar R$.

Summing up, we have proved the following result.

[1] We should work first with the Galerkin approximation $\bar v_a^n$ and then pass to limit as $n \to \infty$. But we show the basic steps for $\bar v_a$.

Proposition 6.1 *Let G be either a linear bounded operator in H satisfying* (3.1) *and with dense range in H or $G = A^\gamma$ with $0 < \gamma < \frac{3}{4}$. Then the Markovian semigroup $\{P_t\}_{t\geq 0}$ is H-irreducible for any $t > 0$.*

Remark 6.2 *If G is diagonal, then all the components have to be non zero, i.e. $Gw(t) = \sum_{j=1}^\infty g_j \beta_j(t) e_j$ with all $g_j \neq 0$ and $\sum_j g_j^2 \lambda_j^{-2} < \infty$.*
Of course, the result holds also for $G = LA^\gamma$ with $\gamma < \frac{3}{4}$ and L an isomorphism in H.

6.2 Strongly Feller

The Markovian semigroup $\{P_t\}_{t\geq 0}$ is strongly Feller in H at time $t > 0$ if

$$P_t : B_b(H) \to C_b(H).$$

This means that, given $\phi \in B_b(H)$

$$\lim_{y_1 \to y_2} P_t\phi(y_1) = P_t\phi(y_2).$$

If the operator G fulfils the assumptions of Proposition 6.1 below, i.e. if G is regular enough and invertible, it is well-known that the Markovian semigroup for the linear equation (3.3) is strongly Feller for any $t > 0$ (see e.g. [15, 19]). Considering equation (2.8) as a non linear version of (3.3), the Markovian semigroup P_t may have this regularizing effect for $t > 0$. There is a formula for its derivative, providing even more regularity under some assumptions on the non linear part (see [15]). To use it, we need to introduce a modified Galerkin equation; then in the limit we recover our equation and the strongly Feller property will be proved. Our technique is similar to that of [20, 21].

For $R \geq 1$, let θ_R be a C^1-function with bounded derivative such that $\theta_R = 1$ on $[-R, R]$ and $\theta_R = 0$ outside $[-R-1, R+1]$; take $|\theta_R|$ and $|\theta_R'|$ bounded by 1. We consider the modified Galerkin system

$$\begin{cases} du^{n,R} + \left[\nu A^2 u^{n,R} - A u^{n,R} + \theta_R(|u^{n,R}(t)|^2) B_n(u^{n,R})\right] dt = \Pi_n G dw \\ u^{n,R}(0) = \Pi_n y \end{cases} \tag{6.4}$$

The cut-off function θ_R introduces minor changes with respect to the Galerkin equation for (2.8); it is straightforward to obtain, as in Section 3, that there exists a unique process $u^{n,R} \in C([0,T]; H)$ solving (6.4).

First we show that the Markovian semigroup $\{P_t^{n,R}\}_{t\geq 0}$ of (6.4) is Lipschitz Feller for $t > 0$. Let $[DP_t^{n,R}\phi(y)] \cdot h$ be the derivative, in direction h, of the mapping $y \mapsto P_t^{n,R}\phi(y)$. Then, Bismut-Elworthy-Li formula represents it by means of an expression depending on $[Du^{n,R}(t;y)] \cdot h$, the derivative, in the direction h, of the mapping $y \mapsto u^{n,R}(t;y)$. Indeed (see [15] and references therein)

$$[DP_t^{n,R}\phi(y)]\cdot h = \frac{1}{t}\mathbb{E}\left[\phi(u^{n,R}(t;y)) \int_0^t \langle (\Pi_n G G^* \Pi_n)^{-\frac{1}{2}} [D\, u^{n,R}(s;y)] \cdot h, \Pi_n dw(s)\rangle\right] \tag{6.5}$$

for all $h \in H_n$. Therefore

$$\left|D\, P_t^{n,R}\phi(y)\cdot h\right| \le \frac{1}{t}\|\phi\|_0 \left[\mathbb{E}\int_0^t \left|(\Pi_n GG^*\Pi_n)^{-\frac12} D\, u^{n,R}(s;y)\cdot h\right|^2 ds\right]^{\frac12}.$$

Here $\|\cdot\|_0$ denotes the supremum norm in C_b (or B_b).
Assuming that G is a linear bounded operator in H with $R(G) \supseteq D(A)$, we have that

$$\int_0^t \left|(\Pi_n GG^*\Pi_n)^{-\frac12} D\, u^{n,R}(s;y)\cdot h\right|^2 ds \le C\int_0^t \left|AD\, u^{n,R}(s;y)\cdot h\right|^2 ds$$

(see details in [20]). This holds also if $G = LA^\gamma$ with $0 < \gamma < \frac34$ as in Remark 3.1.
Estimate on the latter quantity are easily obtained; by (6.4), the equation for $U^{n,R} := D\, u_n^R(\cdot;y)\cdot h$ is

$$\begin{cases} \frac{d}{dt}U^{n,R} + \nu A^2 U^{n,R} - AU^{n,R} + 2\theta_R'(|u^{n,R}|^2)\langle u^{n,R}, U^{n,R}\rangle B_n(u^{n,R}) \\ \qquad\qquad\qquad + \theta_R(|u^{n,R}|^2)[B_n(u^{n,R}, U^{n,R}) + B_n(U^{n,R}, u^{n,R})] = 0, \\ U^{n,R}(0) = \Pi_n h. \end{cases}$$

We multiply scalarly in H the first equation by $U^{n,R}$; by means of relationships for the trilinear forms already used before, we obtain

$$\frac12\frac{d}{dt}|U^{n,R}|^2 + \nu|AU^{n,R}|^2 \le \frac{\nu}{2}|AU^{n,R}|^2 + c_\nu\left[1 + (\theta_R'|u^{n,R}|^3)^2 + (\theta_R|u^{n,R}|)^2\right]|U^{n,R}|^2.$$

Then

$$\frac{d}{dt}|U^{n,R}|^2 + \nu|AU^{n,R}|^2 \le \tilde{C}|U^{n,R}|^2$$

for some constant $\tilde{C}$ dependent on ν but not on R or n.

By Gronwall lemma,

$$\mathbb{E}|U^{n,R}(t)|^2 \le |h|^2 e^{\tilde{C}t}$$

and, integrating in time,

$$\mathbb{E}\int_0^t |AU^{n,R}(s)|^2 ds \le \frac{1}{\nu}|h|^2(1 + e^{\tilde{C}t}). \tag{6.6}$$

Summing up, we have shown that

$$\left|D\, P_t^{n,R}\phi(y)\cdot h\right| \le \frac{1}{t}\|\phi\|_0 c_\nu (1 + e^{\tilde{C}t})^{1/2}|h| =: L_{R,t}\|\phi\|_0|h|.$$

Thus, the derivative $D\, P_t^{n,R}\phi(y)$ is uniformly bounded and by the mean value Theorem

$$\left|P_t^{n,R}\phi(y_1) - P_t^{n,R}\phi(y_2)\right| \le \Big(\sup_{\substack{k,h\in H_n\\ |h|\le 1}} \left|DP_t^{n,R}\phi(k)\cdot h\right|\Big)\ |y_1 - y_2| \le L_{R,t}\|\phi\|_0\ |y_1 - y_2|,$$

that is $P_t^{n,R}\phi$ is Lipschitz Feller for all $t > 0$.

Passing to the limit, as $n \to \infty$, we obtain as usual that, $\mathbb{P}$-a.s., $u^{n,R}$ converges strongly to u^R in $L^2(0,T;H)$, where u^R solves the equation

$$du^R + \left[\nu A^2 u^R - Au^R + \theta_R(|u^R|^2)B(u^R)\right] dt = Gdw$$

Passing to a subsequence, $u^{n,R}(t)$ converges to $u^R(t)$ in H, for a.e. t. Then, for $\phi \in C_b(H)$, a subsequence of $P_t^{n,R}\phi(y)$ converges towards $P_t^R\phi(y)$, for a.e. t. But the trajectories of u^R are continuous in time with values in H and therefore we conclude that for any $t > 0$, $R \geq 1$, $\phi \in C_b(H)$ and $y_1, y_2 \in H$ there exists a constant $L_{R,t}$ depending only on R and t, such that

$$\left|P_t^R\phi(y_1) - P_t^R\phi(y_2)\right| \leq L_{R,t}\|\phi\|_0\, |y_1 - y_2|.$$

The same result holds for $\phi \in B_b(H)$ (see Lemma 7.1.5 in [15]).

The last step consists in letting $R \to \infty$. Working as we did for v_n obtaining (3.8), we can easily verify that

$$\sup_{|y|\leq M} \sup_{0\leq t\leq T} |u(t;y)| < \infty \qquad \mathbb{P} - a.s.$$

and similarly for $u^R(t;y)$. Moreover, the processes u and u^R coincide until u^R lies in the ball of radius R in H. Therefore, given $t > 0$ and $y \in H$, for $\mathbb{P}$-a.e. ω there exists R_ω such that $u^R(t;y)(\omega) = u(t;y)(\omega)$ for all $R \geq R_\omega$, uniformly in y in bounded sets of H. So, given t and $\phi \in B_b(H)$, $\mathbb{P}$-a.s.

$$\lim_{R\to\infty} \phi(u^R(t;y)) = \phi(u(t;y))$$

uniformly in y in bounded sets of H. Since ϕ is bounded, we get that the convergence holds when we take expectation. Hence, for any $t > 0$ and $\phi \in B_b(H)$

$$\lim_{R\to\infty} P_t^R\phi(y) = P_t\phi(y)$$

uniformly in y in bounded sets of H.

Finally, given $t > 0$, $\phi \in B_b(H)$ and $y_1, y_2 \in H$

$$\begin{aligned}\lim_{y_1\to y_2} P_t\phi(y_1) &= \lim_{y_1\to y_2}\lim_{R\to\infty} P_t^R\phi(y_1) = \lim_{R\to\infty}\lim_{y_1\to y_2} P_t^R\phi(y_1) = \lim_{R\to\infty} P_t^R\phi(y_2)\\ &= P_t\phi(y_2).\end{aligned}$$

This proves the strongly Feller property for any $t > 0$. Therefore, we have proved the following result.

Proposition 6.3 *Let G be either a linear bounded operator in H, satisfying* (3.1) *and such that $R(G) \supseteq D(A)$ or $G = A^\gamma$ with $0 < \gamma < \frac{3}{4}$. Then, for any $y \in H$ the Markovian semigroup $\{P_t\}_{t\geq 0}$ is strongly Feller in H for any $t > 0$.*

Remark 6.4 *For instance, $G = A^\gamma$, with $-1 \leq \gamma < \frac{3}{4}$, is an example of operator fulfilling the assumptions of the previous proposition. The case $G = LA^\gamma$, with L an isomorphism in H, can be treated in the same way.*

6.3 Regularity results

In this section we work with the operator G of the form A^γ, considering the values $\gamma < -1$ not included in the previous section (see Remark 6.4). Even if the problem interesting from the physical point of view (that with $\gamma = \frac{1}{2}$) has been solved in the previous sections, we want to show that the limitation $\gamma \geq -1$ can be removed so to prove existence and uniqueness of the invariant measure when $G = A^\gamma$ for any $\gamma < \frac{3}{4}$. The case LA^γ, where L is an isomorphism in H, can be treated in the same way.

We prove the following result.

Proposition 6.5 *Let $G = A^\gamma$ with $\gamma < -1$. For any α such that*

$$1 \leq \alpha < \frac{3}{4} - \gamma, \tag{6.7}$$

given $y \in D(A^\alpha)$ there exists a unique solution u to problem (2.8) *and*

$$u \in C([0,T]; D(A^\alpha)) \ \mathbb{P} - a.s.$$

Proof. Existence and uniqueness hold in the bigger space $C([0,T]; H)$, as proved in Theorem 3.4. Thus, we need to prove the regularity $C([0,T]; D(A^\alpha))$. From (3.10), which now reads $\sum_j \lambda_j^{2(\gamma+\alpha-1)} < \infty$, we obtain that if $\alpha < \frac{3}{4} - \gamma$ the linear equation has solution z_a with paths in $C([0,T]; D(A^\alpha))$. What remain to be proved is that equation (3.5) has a solution v_a with paths in $C([0,T]; D(A^\alpha))$. We show a priori estimates as in the proof of Proposition 3.3. We take the scalar product of this equation with $A^{2\alpha} v_a$, use (2.7) and Young inequality to obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}&|A^\alpha v_a|^2 + \nu |A^{1+\alpha} v_a|^2 \\
&= \langle A^\alpha v_a, A^{1+\alpha} v_a\rangle - \langle A^{\alpha-1} B(v_a + z_a), A^{1+\alpha} v_a\rangle + a\langle A^\alpha z_a, A^\alpha v_a\rangle \\
&\leq |A^\alpha v_a||A^{1+\alpha} v_a| + |A^{\alpha-\frac{1}{2}}(v_a + z_a)|^2 |A^{1+\alpha} v_a| + a|A^\alpha z_a||A^\alpha v_a| \\
&\leq \frac{\nu}{2}|A^{1+\alpha} v_a|^2 + \frac{c_\nu}{2}[|A^\alpha v_a|^2 + |A^\alpha z_a|^2 + |A^{\alpha-\frac{1}{2}}(v_a + z_a)|^4].
\end{aligned}$$

Hence

$$\frac{d}{dt}|A^\alpha v_a|^2 + \nu |A^{1+\alpha} v_a|^2 \leq c_\nu |A^\alpha v_a|^2 + c_\nu [|A^\alpha z_a|^2 + |A^{\alpha-\frac{1}{2}}(v_a + z_a)|^4]. \tag{6.8}$$

First, consider $\alpha = 1$; then

$$\frac{d}{dt}|A v_a|^2 + \nu |A^2 v_a|^2 \leq c_\nu |A v_a|^2 + c_\nu [|A z_a|^2 + |A^{\frac{1}{2}}(v_a + z_a)|^4].$$

But $|A^{\frac{1}{2}}(v_a + z_a)|^4 = |A^{\frac{1}{2}} u|^4$ and it belongs to $L^1(0,T)$; indeed, by Corollary 3.5 (ii) we know that $u \in C([0,T]; H) \cap L^2(0,T; D(A))$ and by interpolation $u \in L^4(0,T; D(A^{\frac{1}{2}}))$. Since $|A z_a|^2 + |A^{\frac{1}{2}}(v_a + z_a)|^4$ is integrable in time, by Gronwall lemma we get as usual that

$$\sup_{0\leq t\leq T} |A v_a(t)| < \infty, \qquad \int_0^T |A^2 v_a(t)|^2 dt < \infty.$$

We do not consider the Galerkin approximations but we should do, as in Section 3, to conclude that

$$v_a \in C([0,T]; D(A)) \cap L^2(0,T; D(A^2)).$$

All the results hold pathwise, as before, and we do not specify this every time.

Coming back to (6.8), if $1 < \alpha \leq \frac{3}{2}$, then

$$\frac{d}{dt}|A^\alpha v_a|^2 + \nu |A^{1+\alpha} v_a|^2 \leq c_\nu |A^\alpha v_a|^2 + c_\nu [|A^\alpha z_a|^2 + |A(v_a + z_a)|^4]$$

From the case $\alpha = 1$ we know that $u = v_a + z_a \in C([0,T]; D(A))$ and therefore $|A^\alpha z_a|^2 + |A(v_a + z_a)|^4$ belongs to $L^1(0,T)$. Once more, Gronwall lemma allows to conclude that

$$\sup_{0 \leq t \leq T} |A^\alpha v_a(t)| < \infty\,, \qquad \int_0^T |A^{1+\alpha} v_a(t)|^2 dt < \infty$$

for $1 < \alpha \leq \frac{3}{2}$ and therefore $v_a \in C([0,T]; D(A^\alpha)) \cap L^2(0,T; D(A^{1+\alpha}))$.

By induction, we prove the result for all $\alpha \geq 1$. Assume that for some integer $m > 2$, given $\frac{m}{2} < \alpha \leq \frac{m+1}{2}$ we have $v_a \in C([0,T]; D(A^\alpha)) \cap L^2(0,T; D(A^{1+\alpha}))$. Then we want to show the regularity result for $\frac{m+1}{2} < \alpha \leq \frac{m+2}{2}$. First, for $\alpha \leq \frac{m+2}{2}$ the term $|A^{\alpha - \frac{1}{2}}(v_a + z_a)|^4$ in (6.8) is bounded by $|A^{\frac{m+1}{2}}(v_a + z_a)|^4$. But, by the induction assumption we know in particular that $u = v_a + z_a \in C([0,T]; D(A^{\frac{m+1}{2}}))$. Hence, as before, we have the suitable a priori estimate

$$\frac{d}{dt}|A^\alpha v_a|^2 + \nu |A^{1+\alpha} v_a|^2 \leq c_\nu |A^\alpha v_a|^2 + c_\nu [|A^\alpha z_a|^2 + |A^{\frac{m+1}{2}}(v_a + z_a)|^4]$$

for $\frac{m+1}{2} < \alpha \leq \frac{m+2}{2}$, which allows to conclude by Gronwall lemma that

$$\sup_{0 \leq t \leq T} |A^\alpha v_a(t)| < \infty\,, \qquad \int_0^T |A^{1+\alpha} v_a(t)|^2 dt < \infty$$

so that $v_a \in C([0,T]; D(A^\alpha)) \cap L^2(0,T; D(A^{1+\alpha}))$ for $\frac{m+1}{2} < \alpha \leq \frac{m+2}{2}$. □

Now, with the same estimates we can prove as in Section 6.2 that the Markovian semigroup is strongly Feller in $D(A^\alpha)$, with α specified by (6.7). The only differences with respect to Section 6.2 are that the cut-off function in (6.4) is $\theta_R(|A^\alpha u|^2)$ instead of $\theta_R(|u|^2)$ and we require $R(G) \supseteq D(A^{1+\alpha})$ instead of $R(G) \supseteq D(A)$. This condition on the range of $G = A^\gamma$ is satisfied if

$$1 + \alpha \geq -\gamma.$$

Moreover, we notice that Proposition 6.5 implies that the Markovian semigroup associated to equation (2.8) is irreducible in $D(A^\alpha)$. Indeed, in Proposition 6.1 we proved, for any $\gamma < \frac{3}{4}$, irreducibility in H. On the other hand, by Proposition 6.5 we know that the Markov process u lives in $D(A^\alpha)$ for α specified by (6.7). Hence, the irreducibility in inherited from H, since an open non empty subset of H, when restricted to $D(A^\alpha)$, is an open non empty subset of $D(A^\alpha)$.

Summing up all the conditions on α, we have the following result

Proposition 6.6 *Given $\gamma < -1$, the Markovian semigroup $\{P_t\}$ associated to equation* (2.8) *is irreducible and strongly Feller in $D(A^\alpha)$ for any $t > 0$, if α satisfies*

$$\begin{cases} 1 \leq \alpha < \frac{3}{4} - \gamma & when \ -2 \leq \gamma < -1 \\ -1 - \gamma \leq \alpha < \frac{3}{4} - \gamma & when \ \gamma < -2 \end{cases} \tag{6.9}$$

7 CONCLUSIONS

We collect all the previous results which imply existence of an invariant measure and its uniqueness, as explained in Section 4.

Theorem 7.1 *Let $G = LA^\gamma$ for $\gamma < \frac{3}{4}$ and L an isomorphism in H. Set $E = H$ if $\gamma \geq -1$ and $E = D(A^\alpha)$ if $\gamma < -1$, where α fulfills* (6.9)*. Then equation* (2.8) *has a unique invariant measure μ, concentrated on the space E. All the transition probability measures $P(t, y, \cdot)$ for $y \in E$ and $t > 0$ are equivalent to μ and this measure μ is ergodic:*

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \phi(u(t; y))dt = \int \phi d\mu$$

$\mathbb{P}$-a.s. and for all $\phi \in L^1(E; \mu)$, $y \in E$.

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